Discrete Green's functions for products of regular graphs

Robert B. Ellis*

Department of Mathematics, Texas A&M University,
College Station, TX 77843-3368, USA

Abstract

Discrete Green's functions are the inverses or pseudo-inverses of combinatorial Laplacians. We present compact formulas for discrete Green's functions, in terms of the eigensystems of corresponding Laplacians, for products of regular graphs with or without boundary. Explicit formulas are derived for the cycle, torus, and 3-dimensional torus, as is an inductive formula for the t-dimensional torus with n vertices, from which the Green's function can be completely determined in time $O(t\, n^{2-1/t} \log n)$. These Green's functions may be used in conjunction with diffusion-like problems on graphs such as electric potential, random walks, and chip-firing games or other balancing games.

Key words: discrete Green's function, combinatorial Laplacian, regular graph, torus

1 Introduction

Discrete Green's functions are the inverses or pseudo-inverses of combinatorial Laplacians, which govern diffusion-like problems on graphs such as random walks, electric potential and chip-firing games or other balancing games. Just as Green's functions in the continuous case depend on the domain and boundary conditions, discrete Green's functions are associated with the underlying graph and boundary conditions, if any. Just as in the continuous case, a new set of discrete Green's functions must be determined for each new class of graphs. Certainly, a discrete Green's function can be determined by brute force (pseudo-)inversion of the corresponding Laplacian, but this is no advancement toward compact or closed-form functions.

In this paper, we develop such compact formulas for discrete Green's functions on products of simple graphs with or without boundary. Section 2 presents the necessary definitions and background. Section 3 illustrates these definitions by deriving the Green's function for the cycle. In Section 4, we derive formulas for products of regular graphs with or without boundary. In Section 5, we illustrate the case of products of regular graphs without boundary by deriving the Green's function for the t-dimensional torus and addressing its computational complexity, with explicit Green's functions given for $t = 2$ and $t = 3$. Finally, the relationship between Green's functions and the hitting time of a random walk is explained and illustrated in the case of the 2-torus in Section 6

*Research supported in part by NSF grant DMS-9977354. *Email address:* rellis@math.tamu.edu.

2 Preliminaries

The basic definitions follow those of [4]. Let $\Gamma = (V, E)$ be a simple connected graph. Let $x, y \in V$ be arbitrary vertices. The *adjacency matrix* A is defined by $A(x, y) = \chi(x \sim y)$. The diagonal *degree matrix* D is defined by $D(x, y) = \chi(x = y) \cdot d_x$, where d_x is the degree of x in Γ. The *volume* of a graph is $vol(\Gamma) = \sum_{v \in V} d_v$. The *Laplacian*, $L = D - A$, of Γ is

$$L(x,y) = \begin{cases} d_x, & \text{if } x = y \\ -1, & \text{if } x \sim y \\ 0, & \text{otherwise.} \end{cases}$$

The *normalized Laplacian*, $\mathcal{L} = D^{-1/2} L D^{-1/2}$, is

$$\mathcal{L}(x,y) = \begin{cases} 1, & \text{if } x = y \\ -1/\sqrt{d_x d_y}, & \text{if } x \sim y \\ 0, & \text{otherwise.} \end{cases}$$

The *discrete Laplace* operator Δ is

$$\Delta(x,y) = \begin{cases} 1, & \text{if } x = y \\ -1/d_x, & \text{if } x \sim y \\ 0, & \text{otherwise.} \end{cases}$$

In general, the following relations hold between L, $\mathcal{L}$, and Δ:

$$\begin{aligned} \mathcal{L} &= D^{-1/2} L D^{-1/2} \\ \mathcal{L} &= D^{1/2} \Delta D^{-1/2} \\ L &= D\Delta. \end{aligned}$$

For $S \subseteq V$, we define the *Dirichlet* versions of L_S, $\mathcal{L}_S$, and Δ_S as the results of deleting the rows and columns corresponding to $V \setminus S$ from L, $\mathcal{L}$, and Δ, respectively. Without loss of generality, we assume that both Γ and the subgraph generated by S to be connected. By abuse of notation, we use S to refer to the subgraph generated by S in Γ. Additionally, we say that the orthonormal eigensystem of S is the orthonormal eigensystem $\{(\lambda_j, \phi_j) : j \in J\}$ of the real symmetric matrix $\mathcal{L}_S$; where $J = \{0, \ldots, |V| - 1\}$ with $\lambda_0 = 0$ when $S = V$, and $J = \{1, \ldots, |S|\}$ with $\lambda_1 > 0$ when $S \neq V$. In either case, we order the eigenvalues by $\lambda_j \leq \lambda_{j+1}$ for all subscripts in range; basic properties such as $0 \leq \lambda_j \leq 2$ for all j are summarized in [3]. In particular, the orthonormal eigenvectors can be chosen to have real entries.

When $S \subsetneq V$, Δ_S, L_S and $\mathcal{L}_S$ are invertible, and the *Green's function* G and *normalized Green's function* $\mathcal{G}$ are determined by the relations

$$\begin{aligned} &\Delta_S G = G \Delta_S = I_S \qquad \text{and} \\ &\mathcal{L}_S \mathcal{G} = \mathcal{G} \mathcal{L}_S = I_S. \end{aligned} \tag{1}$$

We can tie these relations to random walks as follows. Let $P = [P(x, y)]$ be the transition probability matrix for the simple irreducible *transient* random walk on S with absorbing states $V \setminus S$, where the probability p_{xy} of moving to state y from state x is $1/d_x$ if x and y are adjacent and 0 otherwise. Then $\Delta_S = I - P$, and $(I - P)^{-1} = I + P + P^2 + \cdots$ gives

$$G(x,y) = \sum_n P_n(x,y), \tag{2}$$

where $P_n(x, y)$ is the n-step transition probability matrix (cf. [11, p. 31]). See [1, 7] for definitions and results on random walks.

On the other hand, when $S = V$, the sum in (2) does not converge, and we require an alternate definition of the Green's function. Since $\mathcal{L}$ is not invertible, its corresponding normalized Green's function $\mathcal{G}$ is defined by

$$\mathcal{G} = \sum_{\lambda_j > 0} \frac{1}{\lambda_j} \phi_j \phi_j^*. \tag{3}$$

This definition of $\mathcal{G}$ is equivalent to the two relations

$$\begin{aligned} \mathcal{G}\mathcal{L} = \mathcal{L}\mathcal{G} = I - P_0 = I - \phi_0\phi_0^* \qquad & \text{and} \\ \mathcal{G}P_0 = 0, \qquad & \end{aligned} \tag{4}$$

where $P_0 = \phi_0\phi_0^*$ is the projection of the orthonormal eigenvector ϕ_0 corresponding to eigenvalue 0. Since $L\vec{1} = 0$, we have $\phi_0 = D^{1/2}\vec{1}/||D^{1/2}\vec{1}||$, and so $\phi_0(x) = \sqrt{d_x/vol(\Gamma)}$. It is also important to note that when $\mathcal{G}$ is invertible, the definitions in (1) and (3) are equivalent. Furthermore, $\mathcal{G}$ is related to the so-called *fundamental matrix*

$$Z(x, y) := \sum_{n=0}^{\infty} (P_n(x, y) - \pi_y), \tag{5}$$

where π is the stationary distribution of the random walk on Γ, by the equation $\mathcal{G} = D^{1/2}ZD^{-1/2}$; simply verify that $D^{1/2}ZD^{-1/2}$ satisfies (3) and (4). See Chapter 3, p. 17 of [1] for relationships between Z and hitting times. A more complete exposition on discrete Green's functions in the context of spectral graph theory appears in [8].

3 Green's function for the cycle

In this section we illustrate the preceding definitions in the case of the cycle, which has no boundary. The techniques developed here will be used in later sections to construct the Green's functions for higher dimensional tori.

By the cycle C_m, we mean the 2-regular connected graph with vertices $\{0, 1, \ldots, m-1\}$, where $m \geq 3$. The various Laplacians are related by $\Delta = \mathcal{L} = L/2$. Applying the definition in (4), the normalized Green's function $\mathcal{G}$ satisfies

$$\begin{aligned} \mathcal{G}\mathcal{L} \;=\; \mathcal{L}\mathcal{G} \;&=\; I - \frac{1}{m}J \qquad \text{and} \\ \mathcal{G}J \;&=\; 0, \end{aligned} \tag{6}$$

where $\phi_0(x) = \sqrt{1/m}$, and J is the $m \times m$ matrix of 1's. Because the cycle is vertex-transitive, the values $\mathcal{L}(x, y)$ and $\mathcal{G}(x, y)$ depend only on the distance $|y - x|$ between x and y, and the following definition of $\mathcal{G}(a)$ is well-defined:

$$\mathcal{G}(a) := \mathcal{G}(x, y), \qquad \text{if } a = |y - x|. \tag{7}$$

We are ready to derive the Green's function for the cycle.

Theorem 1. *Let $m \geq 3$. For $0 \leq x, y \leq m-1$, the cycle C_m has normalized Green's function*

$$\mathcal{G}(x,y) = \frac{(m+1)(m-1)}{6m} - |y-x| + \frac{(y-x)^2}{m}. \tag{8}$$

Proof. From (6) and (7), we have the recurrence

$$\begin{aligned} 2\mathcal{G}(x,y) - \mathcal{G}(x,y-1) - \mathcal{G}(x,y+1) &= \begin{cases} 2-2/m, & x = y \\ -2/m, & x \neq y, \end{cases} \qquad \text{or} \\ 2\mathcal{G}(a) - \mathcal{G}(a-1) - \mathcal{G}(a+1) &= \begin{cases} 2-2/m, & a = 0, \\ -2/m, & a > 0, \end{cases} \end{aligned}$$

provided that we define $\mathcal{G}(-1) = \mathcal{G}(1)$ for simplicity of representing the case $a = 0$. The following recurrence on differences results:

$$\mathcal{G}(a+1) - \mathcal{G}(a) \quad = \quad \mathcal{G}(a) - \mathcal{G}(a-1) \; + \; \frac{2}{m} \; - \; 2\chi(a=0). \tag{9}$$

The second constraint in (6) determines that the sum of $\mathcal{G}$ across any row must be 0; i.e.,

$$\sum_{a=0}^{m-1} \mathcal{G}(a) \quad = \quad 0. \tag{10}$$

We now solve the recurrence on differences, starting with $\mathcal{G}(1) - \mathcal{G}(0)$ by setting $a = 0$ in (9).

$$\begin{aligned} \frac{1}{2}\left(\mathcal{G}(1) - 2\mathcal{G}(0) + \mathcal{G}(-1)\right) &= \mathcal{G}(1) - \mathcal{G}(0) \\ &= \frac{1}{m} - 1. \end{aligned} \tag{11}$$

Resolving (9) with base case given by (11), we have

$$\begin{aligned} \mathcal{G}(a+1) - \mathcal{G}(a) &= \mathcal{G}(a) - \mathcal{G}(a-1) + \frac{2}{m} \\ &= \mathcal{G}(a-1) - \mathcal{G}(a-2) + 2 \cdot \frac{2}{m} \\ &= \quad \vdots \\ &= \mathcal{G}(1) - \mathcal{G}(0) + a \cdot \frac{2}{m} \\ &= \frac{1}{m} - 1 + a \cdot \frac{2}{m} \; . \end{aligned} \tag{12}$$

Having derived a simple recurrence from the recurrence on differences, we proceed to determine $\mathcal{G}(a)$. Resolving (12) yields

$$\begin{aligned} \mathcal{G}(a) &= \mathcal{G}(a-1) + \frac{1}{m} - 1 + (a-1)\frac{2}{m} \\ &= \mathcal{G}(a-2) + \frac{2}{m} - 2 + [(a-1) + (a-2)]\frac{2}{m} \\ &= \quad \vdots \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{G}(0) + \frac{a}{m} - a + \frac{2}{m}\sum_{k=0}^{a-1} k \\
&= \mathcal{G}(0) - a + \frac{a^2}{m} \ . \qquad (13)
\end{aligned}
$$

Now applying the row sum constraint (10) allows us to compute the value of $\mathcal{G}(0)$.

$$
\begin{aligned}
0 &= \sum_{a=0}^{m-1}\left(\mathcal{G}(0) - a + \frac{a^2}{m}\right) \\
m\cdot\mathcal{G}(0) &= \sum_{a=0}^{m-1}\left(a - \frac{a^2}{m}\right) \\
\mathcal{G}(0) &= \frac{(m+1)(m-1)}{6m} \ . \qquad (14)
\end{aligned}
$$

Plugging (14) into (13) and letting $a = |y-x|$ achieves the desired result. $\square$

Because (3) also gives $\mathcal{G}$ for C_m, we have a whole class of identities formed by choosing any orthonormal eigenbasis for C_m and equating (3) with (8). We give one such well-known basis now, which arises naturally from the consideration of circulant matrices (cf. [6]).

Lemma 1. *For $m \geq 3$, define $\phi_j(x) := \frac{1}{\sqrt{m}}\exp\left(-i\frac{2\pi jx}{m}\right)$. Then $\{\left(1-\cos(2\pi j/m), \phi_j\right) : 0 \leq j < m\}$ is an orthonormal eigensystem of $\mathcal{L}$ for C_m.*

The proof is omitted, but follows from a straightforward verification of the necessary conditions. Theorem 2 follows by combining Theorem 1 with (3) using the orthonormal eigenbasis of Lemma 1.

Theorem 2. *Let $m \geq 3$ and let $0 \leq x, y < m$. Then*

$$
\frac{1}{m}\sum_{j=1}^{m-1}\frac{\exp\left((2\pi ij/m)(y-x)\right)}{1-\cos(2\pi j/m)} = \frac{(m+1)(m-1)}{6m} - |y-x| + \frac{(y-x)^2}{m} \ .
$$

4 Green's functions for products of regular graphs

Theorems 4-5 of [3] give a contour integral formula for the Green's function of the Cartesian product of two regular graphs with boundary, provided a certain generalized Green's function is known for each factor graph. In this section, we extend these results to include the cases where one or both of the factor graphs is without boundary, and provide simplified working formulas requiring the generalized Green's function of one graph and the eigensystem of the other. The original technique of [3] recovers the Green's function of the product graph as the residues of a certain contour integral whose contour contains as poles all of the eigenvalues of one of the graphs, and none of the negatives of the eigenvalues of the other graph. When both graphs are without boundary, both have an eigenvalue of 0, and the resulting order 2 pole requires a modified contour integral formula.

Recalling the definitions in Section 2, Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be regular graphs of degree d and d' with specified vertex subsets S and S' and normalized Dirichlet Laplacians $\mathcal{L}_S$ and

$\mathcal{L}'_{S'}$, respectively. For any $\alpha \in \mathbb{C}$, define $\mathcal{G}_\alpha$ to be the symmetric matrix satisfying the relation $(\mathcal{L}_S + \alpha)\mathcal{G}_\alpha = I_S$, if $S \subsetneq V$; and the relations

$$\begin{aligned}(\mathcal{L}_S + \alpha)\mathcal{G}_\alpha &= I_S - P_0, \qquad \text{and} \\ \mathcal{G}_\alpha P_0 &= 0, \end{aligned} \tag{15}$$

if $S = V$ ($\mathcal{L}_S$ is singular). In either case, this is equivalent to

$$\mathcal{G}_\alpha(x, y) = \sum_{\lambda_j > 0} \frac{1}{\lambda_j + \alpha} \phi_j(x)\overline{\phi_j(y)}, \tag{16}$$

where the ϕ_j's are the orthonormal eigenfunctions of $\mathcal{L}_S$ associated with the eigenvalues λ_j. In particular, $\mathcal{G}_\alpha$ is a rational function of α. The analogous definitions of $\mathcal{G}'_\alpha$, ϕ'_i, and λ'_i are made for Γ'.

The choice of S and S' induces the specified vertex set $S \times S'$ in the Cartesian product $\Gamma \times \Gamma'$. This product of $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ has vertex set $\{(v, v') : v \in V, v' \in V'\}$ and edges of the form $\{(v, v'), (v, u')\}$ or $\{(v, v'), (u, v')\}$ where $\{u, v\} \in E$, $\{u', v'\} \in E'$. Let $\mathcal{L}^\times$ be the normalized Laplacian of $\Gamma \times \Gamma'$, with corresponding normalized Dirichlet Laplacian $\mathcal{L}^\times_{S\times S'}$. We abuse notation by referring to S, S' and $S \times S'$ as the graphs they induce. If $\mathcal{L}_S$ and $\mathcal{L}'_{S'}$ have orthonormal eigensystems $\{(\lambda_j, \phi_j) : j \in J\}$ and $\{(\lambda'_k, \phi'_k) : k \in K\}$, respectively, then when S and S' are regular of the same degree, $\mathcal{L}^\times_{S\times S'}$ has orthonormal eigensystem

$$\left\{ \left(\frac{\lambda_j + \lambda'_k}{2}, \Phi_{j,k} \right) : j \in J, k \in K \right\}, \tag{17}$$

where $\Phi_{j,k}(v, v') := \phi_j(v)\phi'_k(v')$. We begin with the case in which S has boundary.

4.1 At least one graph has boundary

Suppose $S \subsetneq V$, so that S generates a connected subgraph with boundary in Γ. We allow $S' \subseteq V'$ to generate an arbitrary connected subgraph in Γ'. We consider two cases, where the factor graphs are regular of the same degree or regular of different degrees.

First, suppose S and S' be regular of the same degree, and let C denote a simple closed contour in the complex plane consisting of all $\alpha \in \mathbb{C}$ satisfying $|\alpha - 1| = 1 + \lambda_1/2$. The contour C is designed to enclose all of the λ'_k's and none of the $-\lambda_j$'s. We have the following generalization of Theorem 4 of [3].

Theorem 3. *Let S and S' be induced subgraphs of $\Gamma = (V, E)$ and $\Gamma' = (V', E')$, respectively, which are both regular of degree d. Let $S \subsetneq V$ and $S' \subseteq V'$. The normalized Green's function $\mathbf{G}$ of the Cartesian product $S \times S'$ with Dirichlet boundary condition is*

$$\mathbf{G}((x, x'), (y, y')) = \frac{1}{\pi i} \int_C \mathcal{G}_\alpha(x, y)\mathcal{G}'_{-\alpha}(x', y')d\alpha.$$

Proof. Combining the product graph eigensystem in (17) with the formal definition of $\mathbf{G}$ in (3), we have

$$\mathbf{G}((x, x'), (y, y')) \quad = \quad 2\sum_{j,k} \frac{\Phi_{j,k}(x, x')\overline{\Phi_{j,k}(y, y')}}{\lambda_j + \lambda'_k}$$

$$
\begin{aligned}
&= \frac{1}{\pi i}\int_C \sum_{j=1}^{|S|}\sum_k \frac{\phi_j(x)\overline{\phi_j(y)}\phi'_k(x')\overline{\phi'_k(y')}}{(\lambda_j+\alpha)(\lambda'_k-\alpha)}d\alpha \\
&= \frac{1}{\pi i}\int_C \mathcal{G}_\alpha(x,y)\mathcal{G}'_{-\alpha}(x',y')d\alpha, \qquad \text{by (16).}
\end{aligned}
$$

□

Note that the above contour integral picks up exactly the residues at $\alpha = \lambda'_k$. For example, the residue at $\alpha = \lambda'_k$ is exactly

$$
\sum_{\lambda'_K,\lambda'_K=\lambda'_k}\sum_{j=1}^{|S|}\frac{\phi_j(x)\overline{\phi_j(y)}\phi'_K(x')\overline{\phi'_K(y')}}{(\lambda_j+\lambda'_k)} = \sum_{\lambda'_K,\lambda'_K=\lambda'_k}\phi'_K(x')\overline{\phi'_K(y')}\cdot\mathcal{G}_{\lambda'_k}(x,y). \tag{18}
$$

For convenience, we assign the term of the residue in (18) corresponding to K to λ'_K. This observation gives us the computational formula for $\mathbf{G}$ in the following corollary, which in practice may be applied to yield a closed formula for a specific product graph, or to generate in conjunction with (3) a non-trivial identity involving $\mathcal{G}_\alpha$ and arbitrary orthonormal eigensystems of $\mathcal{L}_S$, $\mathcal{L}_{S'}$ and $\mathcal{L}^{\times}_{S\times S'}$.

Corollary 1. *Under the same conditions as in Theorem 3, we have*

$$
\mathbf{G}((x,x'),(y,y')) = 2\sum_k \phi'_k(x')\overline{\phi'_k(y')}\mathcal{G}_{\lambda'_k}(x,y).
$$

Second, suppose we have the same conditions as Theorem 3, except S and S' are regular of degrees d and d', respectively. The normalized Dirichlet Laplacian $\mathcal{L}^{\times}_{S\times S'}$ has orthonormal eigensystem

$$
\left\{\left(\frac{d}{d+d'}\lambda_j + \frac{d'}{d+d'}\lambda'_k, \Phi_{j,k}\right) : j\in J, k\in K\right\}. \tag{19}
$$

The poles of $\mathcal{G}_{\alpha/d}$ are at $\alpha = -d\lambda_j$, and the poles of $\mathcal{G}_{-\alpha/d'}$ are at $\alpha = d'\lambda'_k$. Let C denote a simple closed contour in the complex plane consisting of all $\alpha\in\mathbb{C}$ satisfying $|\alpha - d'| = d' + d\lambda_1/2$; thus C contains all of the $d'\lambda'_k$'s but none of the $-d\lambda_j$'s. We obtain the following minor extension to Theorem 5 of [3].

Theorem 4. *Let S and S' be induced subgraphs of $\Gamma = (V,E)$ and $\Gamma' = (V',E')$, respectively, where Γ is regular of degree d and Γ' is regular of degree d'. Let $S \subsetneq V$ and $S' \subseteq V'$. The normalized Green's function $\mathbf{G}$ of the Cartesian product $S\times S'$ with Dirichlet boundary condition is*

$$
\mathbf{G}((x,x'),(y,y')) = \frac{d+d'}{2\pi i d d'}\int_C \mathcal{G}_{\alpha/d}(x,y)\mathcal{G}'_{-\alpha/d'}(x',y')d\alpha.
$$

Proof. Combining (19) with (3), we have

$$
\begin{aligned}
\mathbf{G}((x,x'),(y,y')) &= \sum_{j,k}\frac{d+d'}{d\lambda_j + d'\lambda'_k}\Phi_{j,k}(x,x')\overline{\Phi_{j,k}(y,y')} \\
&= \frac{d+d'}{2\pi i}\int_C \sum_{j=1}^{|S|}\sum_k \frac{\phi_j(x)\overline{\phi_j(y)}\phi'_k(x')\overline{\phi'_k(y')}}{(d\lambda_j+\alpha)(d'\lambda'_k-\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{d+d'}{2\pi i d d'} \int_C \sum_{j=1}^{|S|} \sum_k \frac{\phi_j(x)\overline{\phi_j(y)}\phi'_k(x')\overline{\phi'_k(y')}}{(\lambda_j + \alpha/d)(\lambda'_k - \alpha/d')} \\
&= \frac{d+d'}{2\pi i d d'} \int_C \mathcal{G}_{\alpha/d}(x,y)\mathcal{G}'_{-\alpha/d'}(x',y')d\alpha.
\end{aligned}
$$

□

Analogous to Corollary 1, by inspecting the residues of the above contour integral at all values $\alpha = d'\lambda'_k$, we have the following.

Corollary 2. *Under the same conditions as Theorem 4, we have*

$$
\mathbf{G}((x,x'),(y,y')) = \frac{d+d'}{d} \sum_k \phi'_k(x')\overline{\phi'_k(y')}\mathcal{G}_{d'\lambda'_k/d}(x,y).
$$

4.2 Neither graph has boundary

Here we consider the case of $S = V$ and $S' = V'$, so that $S \times S'$ is the entire product graph $\Gamma \times \Gamma'$; recall that the normalized Green's function for $\Gamma \times \Gamma'$ is non-invertible. Let $m = |V|$ and $n = |V'|$, and consider first the case in which Γ and Γ' have the same degree.

The orthonormal eigensystems of $\mathcal{L}$ and $\mathcal{L}'$ are $\{(\phi_j, \lambda_j) : 0 \le j \le m-1\}$ and $\{(\phi'_k, \lambda'_k) : 0 \le k \le n-1\}$, respectively. Let C denote a simple closed contour in the complex plane consisting of all $\alpha \in \mathbb{C}$ satisfying $|\alpha - (2 + \lambda'_1/2)| = 2$. The contour C is designed to enclose $\lambda'_1, \ldots, \lambda'_{n-1}$ and none of $-\lambda_0, \ldots, -\lambda_{m-1}$ or λ'_0.

Theorem 5. *Let Γ and Γ' be connected regular graphs of degrees d without boundary. With the notation above, the normalized Green's function $\mathbf{G}$ of the Cartesian product $\Gamma \times \Gamma'$ is*

$$
\mathbf{G}((x,x'),(y,y')) = \frac{1}{\pi i} \int_C \mathcal{G}_\alpha(x,y)\mathcal{G}'_{-\alpha}(x',y')d\alpha + \frac{2}{n}\mathcal{G}(x,y) + \frac{2}{m}\mathcal{G}'(x',y').
$$

Proof. The eigenvector ϕ_0 corresponding to eigenvalue 0 is determined by $\phi_0(x) = \sqrt{(d_x/vol(\Gamma))}$. Combining the formal definition of $\mathbf{G}$ in (3) with (17), and noting that $d_v = d_{v'} = d$, $vol(\Lambda) = d \cdot m$, and $vol(\Lambda') = d \cdot n$, we have

$$
\begin{aligned}
\mathbf{G}((x,x'),(y,y')) &= 2\sum_{j=1}^{m-1}\sum_{k=1}^{n-1} \frac{\Phi_{j,k}(x,x')\overline{\Phi_{j,k}(y,y')}}{\lambda_j + \lambda'_k} \\
&\quad + 2\sum_{j=1}^{m-1} \frac{\Phi_{j,0}(x,x')\overline{\Phi_{j,0}(y,y')}}{\lambda_j} + 2\sum_{k=1}^{n-1} \frac{\Phi_{0,k}(x,x')\overline{\Phi_{0,k}(y,y')}}{\lambda'_k} \\
&= \frac{1}{\pi i} \int_C \sum_{j=1}^{m-1}\sum_{k=1}^{n-1} \frac{\phi_j(x)\overline{\phi_j(y)}\phi_k(x')\overline{\phi_k(y')}}{(\lambda_j + \alpha)(\lambda'_k - \alpha)} d\alpha \\
&\quad + 2\frac{\sqrt{d_{x'}d_{y'}}}{vol(\Gamma')} \sum_{j=1}^{m-1} \frac{\phi_j(x)\overline{\phi_j(y)}}{\lambda_j} + 2\frac{\sqrt{d_x d_y}}{vol(\Gamma)} \sum_{k=1}^{n-1} \frac{\phi_k(x')\overline{\phi_k(y')}}{\lambda'_k}
\end{aligned}
$$

$$= \frac{1}{\pi i}\int_C \mathcal{G}_\alpha(x,y)\mathcal{G}'_{-\alpha}(x',y')d\alpha + \frac{2}{n}\mathcal{G}(x,y) + \frac{2}{m}\mathcal{G}'(x',y') \qquad \text{by (16).}$$

□

Analogous to Corollary 1, inspecting the residues of the above contour integral at $\lambda'_1, \ldots, \lambda'_{n-1}$ yields the following corollary.

Corollary 3. *Under the same conditions as in Theorem 5, we have*

$$\mathbf{G}((x,x'),(y,y')) = 2\sum_{k=1}^{n-1} \phi'_k(x')\overline{\phi'_k(y')}\mathcal{G}_{\lambda'_k}(x,y) + \frac{2}{n}\mathcal{G}(x,y) + \frac{2}{m}\mathcal{G}'(x',y').$$

Now suppose we have the same conditions as Theorem 3, except that the graphs Γ and Γ' are regular of degrees d and d', respectively, and $\mathcal{L}^{\times}_{\Gamma\times\Gamma'}$ has orthonormal eigensystem given by (19). Let C denote a contour in the complex plane consisting of all $\alpha \in \mathbb{C}$ satisfying $|\alpha - (d' + d'\lambda'_1/2)| = d'$. Thus C contains $d'\lambda'_1, \cdots, d'\lambda'_{n-1}$, but neither $-d\lambda_0, \ldots, -d\lambda_{m-1}$ nor $d'\lambda'_0$. We obtain the following theorem.

Theorem 6. *Let Γ and Γ' be connected regular graphs without boundary of degrees d and d', respectively. With the notation above, the normalized Green's function $\mathbf{G}$ of the Cartesian product $\Gamma\times\Gamma'$ is*

$$\mathbf{G}((x,x'),(y,y')) = \frac{d+d'}{2\pi i dd'}\int_C \mathcal{G}_{\alpha/d}(x,y)\mathcal{G}'_{-\alpha/d'}(x',y')d\alpha + \frac{d+d'}{dn}\mathcal{G}(x,y) + \frac{d+d'}{d'm}\mathcal{G}'(x',y').$$

Proof. Combining the definition of $\mathbf{G}$ in (3) with (19), we have

$$\begin{aligned}
\mathbf{G}((x,x'),(y,y')) &= \sum_{j=1}^{m-1}\sum_{k=1}^{n-1} \frac{d+d'}{d\lambda_j + d'\lambda'_k}\Phi_{j,k}(x,x')\overline{\Phi_{j,k}(y,y')} \\
&\quad + \sum_{j=1}^{m-1}\frac{d+d'}{d\lambda_j}\Phi_{j,0}(x,x')\overline{\Phi_{j,0}(y,y')} + \sum_{k=1}^{n-1}\frac{d+d'}{d'\lambda'_k}\Phi_{0,k}(x,x')\overline{\Phi_{0,k}(y,y')} \\
&= \frac{d+d'}{2\pi i dd'}\int_C \frac{dd'\phi_j(x)\overline{\phi_j(y)}\phi_k(x')\overline{\phi_k(y')}}{(d\lambda_j+\alpha)(d'\lambda_k-\alpha)}d\alpha \\
&\quad + \frac{d+d'}{d}\frac{\sqrt{d_{x'}d_{y'}}}{vol(\Gamma')}\mathcal{G}(x,y) + \frac{d+d'}{d}\frac{\sqrt{d_x d_y}}{vol(\Gamma)}\mathcal{G}'(x',y') \\
&= \frac{d+d'}{2\pi i dd'}\int_C \mathcal{G}_{\alpha/d}(x,y)\mathcal{G}'_{-\alpha/d'}(x',y')d\alpha + \frac{d+d'}{dn}\mathcal{G}(x,y) + \frac{d+d'}{d'm}\mathcal{G}'(x',y').
\end{aligned}$$

□

Analogous to Corollary 1, by inspecting the residues of the contour integral at $d'\lambda'_1, \ldots, d'\lambda'_{n-1}$, we have the following.

Corollary 4. *Under the same conditions as in Theorem 6, we have*

$$\mathbf{G}((x,x'),(y,y')) = \frac{d+d'}{d}\sum_{k=1}^{n-1}\phi'_k(x')\overline{\phi'_k(y')}\mathcal{G}_{d'\lambda'_k/d}(x,y) + \frac{d+d'}{dn}\mathcal{G}(x,y) + \frac{d+d'}{d'm}\mathcal{G}'(x',y').$$

5 Example Green's functions for products of cycles

Application of the results in the previous two sections to specific examples is limited only to cases in which the necessary raw materials can be computed. The results in Section 4.1 for the product where at least one graph has boundary require the eigensystem of one graph and the generalized Green's function $\mathcal{G}_\alpha$ of the other graph. Although as written, the results assume that the generalized Green's function is known for the graph without boundary, and the eigensystem is known for the graph without boundary, Theorems 3 and 4 can be easily re-derived for when the reverse is true.

The results in Section 4.2 for the product of two graphs without boundary require knowledge of the eigensystem of one graph, the generalized Green's function $\mathcal{G}_\alpha$ of the other graph, and the Green's functions $\mathcal{G}$ and $\mathcal{G}'$ of both graphs. Thus computation of the normalized Green's function for any graph can be done via any decomposition of the graph into factor graphs where this information is known. This observation can be particularly useful in computing the normalized Green's function inductively where each additional factor graph is from a specific family.

5.1 The torus $C_m \times C_n$

Following Corollary 3, determination of the Green's function of the torus requires the Greens function $\mathcal{G}$ and generalized Green's function $\mathcal{G}_\alpha$ of the cycle. In obtaining a compact formula for the torus, it is critical to simplify $\mathcal{G}_\alpha$ as much as possible before incorporating it into Corollary 3.

Theorem 7. *Let $m, n \geq 3$. For $0 \leq x, y \leq m-1$ and $0 \leq x', y' \leq n-1$, the torus $C_m \times C_n$ has normalized Green's function*

$$\begin{aligned}\mathbf{G}((x,x'),(y,y')) = {} & \frac{2}{n}\sum_{k=1}^{n-1} \exp\left((2\pi i k/n)(y'-x')\right)\Bigg[-\frac{1}{m(1-\cos(2\pi k/n))} \\ & + \frac{T_{m/2-|y-x|}(2-\cos(2\pi k/n))}{(1-\cos(2\pi k/n))(3-\cos(2\pi k/n))U_{m/2-1}(2-\cos(2\pi k/n))}\Bigg] \\ & + \frac{2}{n}\left(\frac{(m+1)(m-1)}{6m} - |y-x| + \frac{(y-x)^2}{m}\right) \\ & + \frac{2}{m}\left(\frac{(n+1)(n-1)}{6n} - |y'-x'| + \frac{(y'-x')^2}{n}\right)\end{aligned}$$

where T and U are the Chebyshev polynomials of the first and second kind, respectively.

Note that the formula depends only on the distances between y and x and between y' and x', which is expected due to the translational symmetries of the torus. The first step of the proof is to obtain a closed form for $\mathcal{G}_\alpha$ for the cycle C_m ($\mathcal{G}$ was determined in Theorem 1). The proof of Theorem 7 is deferred until after Cor. 5.

Theorem 8. *Let $m \geq 3$. For a cycle C_m with vertices $0, 1, \ldots, m-1$, complex $\alpha \neq 0$, and $0 \leq x, y \leq m-1$, the generalized Green's function $\mathcal{G}_\alpha$ satisfies*

$$\mathcal{G}_\alpha(x,y) \;=\; -\frac{2}{m\,(r+r^{-1}-2)} + \frac{2(r^{m/2-|x-y|}+r^{-m/2+|x-y|})}{(r-r^{-1})(r^{m/2}-r^{-m/2})},$$

where $2(1+\alpha) = r + r^{-1}$.

Proof. Because C_m is vertex-transitive, $\mathcal{G}_\alpha(x,y)$ depends only on the distance $\min(|y-x|, m-|y-x|)$ between x and y. Therefore define $\mathcal{G}_\alpha(a) := \mathcal{G}_\alpha(x,y)$, where $a = |y-x|$; this induces the additional relation $\mathcal{G}_\alpha(a) = \mathcal{G}_\alpha(m-a)$ for $1 \le a \le m-1$. From (15), we have

$$\begin{aligned}
\chi(x=y) - \frac{1}{m} &= (\mathcal{L}_S + \alpha)\,\mathcal{G}_\alpha(x,y) \\
&= \frac{1}{2}\left(2(1+\alpha)\mathcal{G}_\alpha(x,y) - \mathcal{G}_\alpha(x+1,y) - \mathcal{G}_\alpha(x-1,y)\right) \\
&= \frac{1}{2}\left((r+r^{-1})\mathcal{G}_\alpha(x,y) - \mathcal{G}_\alpha(x+1,y) - \mathcal{G}_\alpha(x-1,y)\right),
\end{aligned}$$

where $\mathcal{L}_S$ is the normalized Laplacian of C_m. We can rewrite this as

$$\mathcal{G}_\alpha(x-1,y) - r\mathcal{G}_\alpha(x,y) = \frac{2}{m} - 2\chi(x=y) + \frac{1}{r}(\mathcal{G}_\alpha(x,y) - r\mathcal{G}_\alpha(x+1,y)),$$

which for $a > 0$ becomes

$$\begin{aligned}
\mathcal{G}_\alpha(a+1) - r\,\mathcal{G}_\alpha(a) &= \frac{2}{m} + \frac{1}{r}\left(\mathcal{G}_\alpha(a) - r\,\mathcal{G}_\alpha(a-1)\right) \\
&= \quad \vdots \\
&= \frac{2}{m} + \cdots + \frac{1}{r^{a-1}}\frac{2}{m} + \frac{1}{r^a}\left(\mathcal{G}_\alpha(1) - r\,\mathcal{G}_\alpha(0)\right). \qquad (20)
\end{aligned}$$

When $a = 0$,

$$\mathcal{G}_\alpha(1) - r\,\mathcal{G}_\alpha(0) = \frac{2}{m} - 2 + \frac{1}{r}\left(\mathcal{G}_\alpha(0) - r\,\mathcal{G}_\alpha(m-1)\right),$$

and since $\mathcal{G}_\alpha(1) = \mathcal{G}_\alpha(m-1)$, we obtain

$$\begin{aligned}
\mathcal{G}_\alpha(1) &= \frac{1}{m} - 1 + \frac{r+r^{-1}}{2}\,\mathcal{G}_\alpha(0) \\
\mathcal{G}_\alpha(1) - r\mathcal{G}_\alpha(0) &= \frac{1}{m} - 1 + \frac{-r+r^{-1}}{2}\,\mathcal{G}_\alpha(0) \qquad (21)
\end{aligned}$$

From this point, the reader who wishes to verify details is encouraged to employ any standard computer algebra system. Equations (20) and (21) define a recurrence with an initial condition on differences of $\mathcal{G}_\alpha$, which is resolved by substituting $\mathcal{G}_\alpha(1) - r\mathcal{G}_\alpha(0)$ from (21) into (20) and simplifying the geometric series. For $a \ge 0$ this yields

$$\mathcal{G}_\alpha(a+1) - r\,\mathcal{G}_\alpha(a) = \frac{2}{r^{a-1}m}\frac{1-r^a}{1-r} + \frac{1}{r^a}\left(\frac{1}{m} - 1 + \frac{-r+r^{-1}}{2}\,\mathcal{G}_\alpha(0)\right). \qquad (22)$$

Denoting the right-hand side of (22) by $D_\alpha(a)$, for $a > 0$ we have

$$\begin{aligned}
\mathcal{G}_\alpha(a) &= r\,\mathcal{G}_\alpha(a-1) + D_\alpha(a-1) \\
&= \quad \vdots \\
&= r^a\mathcal{G}_\alpha(0) + r^{a-1}D_\alpha(0) + r^{a-2}D_\alpha(1) + \cdots + r^0 D_\alpha(a-1). \qquad (23)
\end{aligned}$$

A careful but straightforward summing of geometric series in (23) yields, for $a > 0$,

$$\mathcal{G}_\alpha(a) = \frac{1}{2}\,\mathcal{G}_\alpha(0)\frac{1+r^{2a}}{r^a}$$

$$+\frac{2}{m}\frac{1}{r^{a-2}}\frac{1-r^a}{1-r}\left(\frac{1+r^a}{1-r^2}+\frac{r^{a-1}}{1-r}+\frac{1+r^a}{1+r}\frac{1-m}{2r}\right). \tag{24}$$

Now using (24), we set $\mathcal{G}_\alpha(1)=\mathcal{G}_\alpha(m-1)$ and solve for $\mathcal{G}_\alpha(0)$, obtaining

$$\mathcal{G}_\alpha(0)=-\frac{2}{m}\frac{r}{(r-1)^2}+\frac{2r(1+r^m)}{(r^2-1)(r^m-1)}, \tag{25}$$

which together with (24) and simplification yields

$$\begin{aligned}\mathcal{G}_\alpha(a) &= \mathcal{G}_\alpha(0)-\frac{2\left(r^{a/2}-r^{-a/2}\right)\left(r^{m/2-a/2}-r^{-(m/2-a/2)}\right)}{\left(r-r^{-1}\right)\left(r^{m/2}-r^{-m/2}\right)}\\ &= \frac{-2}{m\left(r+r^{-1}-2\right)}+\frac{2\left(r^{m/2}+r^{-m/2}\right)}{\left(r-r^{-1}\right)\left(r^{m/2}-r^{-m/2}\right)}\\ &\quad -\frac{2\left(r^{a/2}-r^{-a/2}\right)\left(r^{m/2-a/2}-r^{-(m/2-a/2)}\right)}{\left(r-r^{-1}\right)\left(r^{m/2}-r^{-m/2}\right)}\\ &= \frac{-2}{m\left(r+r^{-1}-2\right)}+\frac{2(r^{m/2-|x-y|}+r^{-m/2+|x-y|})}{(r-r^{-1})(r^{m/2}-r^{-m/2})}.\end{aligned}$$

Substituting $|y-x|$ for a gives the desired formula for $\mathcal{G}_\alpha$. □

By the definition of α and r, we may use the substitution $r=e^{i\theta}$ to rewrite $(r^z+r^{-z})/2=\cos z\theta$ and $(r^z-r^{-z})/2i=\sin z\theta$. Together with the definition of the Chebyshev polynomials of the first and second kinds, T_n and U_n, respectively; i.e.,

$$T_n(x):=\cos n\theta \quad \text{and}$$
$$U_n(x):=\frac{\sin{(n+1)\theta}}{\sin\theta},$$

where $x=\cos\theta$, we obtain the following corollary to Theorem 8.

Corollary 5. *Let $m\geq 3$. For complex $\alpha\neq 0$ and $0\leq x,y\leq m-1$, the generalized Green's function $\mathcal{G}_\alpha$ for the cycle C_m with vertices $0,1,\ldots,m-1$ satisfies*

$$\mathcal{G}_\alpha(x,y) \quad = \quad -\frac{1}{m\alpha}+\frac{T_{m/2-|y-x|}(1+\alpha)}{\alpha(2+\alpha)U_{m/2-1}(1+\alpha)}$$

where T and U are the Chebyshev polynomials of the first and second kinds, respectively.

Proof of Theorem 7: The theorem follows by applying the integral formula for products of graphs without boundary in Corollary 3 to the torus, where $\Gamma=C_m$, $\Gamma'=C_n$, the ϕ''s are the orthonormal basis described in Lemma 1, $\mathcal{G}$ and $\mathcal{G}'$ are given by Theorem 1, and $\mathcal{G}_\alpha$ is given by Corollary 5. □

Combining Theorem 7 with (3) using the orthonormal eigensystem of Lemma 1 for both C_m and C_n yields the following nontrivial identity.

Corollary 6. *Let $m, n \geq 3$; $0 \leq x, y \leq m-1$; and $0 \leq x', y' \leq n-1$. Then*

$$\begin{aligned}
&\frac{1}{mn} \sum_{(j,k)\neq(0,0)} \frac{\exp\big((2\pi ij/m)(y-x)\big)\exp\big((2\pi ik/n)(y'-x')\big)}{\big(1-\cos(2\pi j/m)/2-\cos(2\pi k/n)/2\big)} \\
&\qquad = \frac{2}{n}\sum_{k=1}^{n-1} \exp\big((2\pi ik/n)(y'-x')\big)\Big[-\frac{1}{m(1-\cos(2\pi k/n))} \\
&\qquad + \frac{T_{m/2-|y-x|}(2-\cos(2\pi k/n))}{(1-\cos(2\pi k/n))(3-\cos(2\pi k/n))U_{m/2-1}(2-\cos(2\pi k/n))}\Big] \\
&\qquad + \frac{(m+1)(m-1)}{3mn} - \frac{|y-x|}{n} + \frac{(y-x)^2}{mn} \\
&\qquad + \frac{(n+1)(n-1)}{3mn} - \frac{|y'-x'|}{m} + \frac{(y'-x')^2}{mn},
\end{aligned}$$

where T and U are the Chebyshev polynomials of the first and second kinds, respectively.

The Laplacian of C_m has a 1-dimensional eigenspace corresponding to eigenvalue 0, and a second 1-dimensional eigenspace corresponding to eigenvalue 2 iff C_m is bipartite (when m is even). Otherwise, all eigenspaces are 2-dimensional, since $\lambda_j = \lambda_{m-j}$ for all $1 \leq j \leq m-1$. This means that Corollary 6 is only one of a class of identities constructed by choosing orthonormal eigensystems for C_m and C_n for the left-hand side, and a possibly distinct orthonormal eigensystem for C_n on the right-hand side.

5.2 The t-dimensional torus, $C_{m_1} \times C_{m_2} \times \cdots \times C_{m_t}$

The bottleneck in computing the normalized Green's function for the t-torus via Corollary 3 is the lack of a formula for the generalized Green's function for any $t \geq 2$. Thus the decomposition into a cycle, for which $\mathcal{G}_\alpha$ is given by Corollary 5, and a $(t-1)$-torus is required.

Before giving the normalized Green's function of $C_{m_1} \times \cdots \times C_{m_t}$, we present the information on the components still needed. Choose and label the eigensystem of each C_{m_s} by

$$\{(\lambda_{j_s}^{(s)}, \phi_{j_s}^{(s)}) : 0 \leq j_s \leq m_s - 1\},$$

where $m_s \geq 3$, $0 = \lambda_0^{(s)} < \lambda_1^{(s)} \leq \cdots \leq \lambda_{m_s-1}^{(s)}$, and the vectors $\{\phi_{j_s}^{(s)} : 0 \leq j_s \leq m_s - 1\}$ are orthonormal. The eigenvalues of $C_{m_2} \times \cdots \times C_{m_t}$ are averages of the eigenvalues of the factors C_{m_s}, and the corresponding eigenvectors are products of the eigenvectors of the factors. This is summarized in the next well-known lemma, whose proof is a straightforward induction on (17).

Lemma 2. *The eigenvalues of $C_{m_2} \times \cdots \times C_{m_t}$ are*

$$\Lambda_{j_2,\ldots,j_t} \;=\; \frac{\lambda_{j_2}^{(2)} + \cdots + \lambda_{j_t}^{(t)}}{t-1},$$

where $0 \leq j_s \leq m_s - 1$ for all $2 \leq s \leq t$, with corresponding eigenvectors

$$\Phi_{j_2,\ldots,j_t}(x_2, \ldots, x_t) \;:=\; \prod_{s=2}^{t} \phi_{j_s}^{(s)}(x_s).$$

For the following theorem, let $\mathcal{G}$ be the normalized Green's function for C_{m_1} from Theorem 1, and $\mathcal{G}'$ the normalized Green's function for $C_{m_2} \times \cdots C_{m_t}$.

Theorem 9. *Let $t \geq 2$. Let $0 \leq x_{j_s}, y_{j_s} \leq m_s - 1$ where $m_s \geq 3$ for $1 \leq s \leq t$. The t-dimensional torus $C_{m_1} \times \cdots \times C_{m_t}$ has normalized Green's function*

$$\begin{aligned}\mathbf{G}((x_1,\ldots,x_t),(y_1,\ldots,y_t)) \ &= \ t \sum_{K\neq(0,\ldots,0)} \Phi_K(x_2,\ldots,x_t)\overline{\Phi_K(y_2,\ldots,y_t)}\mathcal{G}_{\Lambda_K}(x_1,y_1) \\ &+ \frac{t}{(t-1)m_1}\mathbf{G}'((x_2,\ldots,x_t),(y_2,\ldots,y_t)) + \frac{t}{m_2\cdots m_t}\mathcal{G}(x_1,y_1), \end{aligned} \tag{26}$$

where K ranges over all indices $(j_2,\ldots,j_t) \neq (0,\ldots,0)$, Φ and Λ are defined in Lemma 2, $\mathcal{G}_\alpha$ is given by Corollary 5, and $\mathbf{G}'$ is the normalized Green's function for $C_{m_2} \times \cdots \times C_{m_t}$.

Proof. The proof proceeds by using $\Gamma = C_{m_1}$ and $\Gamma' = C_{m_2} \times \cdots \times C_{m_t}$ in Corollary 4. The degree of Γ is $d = 2$, and the degree of Γ' is $d' = 2(t-1)$. The result follows. $\square$

Although $\mathbf{G}'$ in Theorem 9 may not already be known, it can be computed inductively from repeated applications of the theorem. Determination of $\mathcal{G}_\alpha$ for any small product $C_1 \times \cdots \times C_{t'}$ would allow the reduction in the number of applications required by a factor of t'. The following formula for the 3-torus is obtained by two applications of Theorem 9, first taking the product of $\Gamma = C_m$ with $\Gamma' = C_m \times C_m$, and then the product of $\Gamma = C_m$ with $\Gamma' = C_m$.

Corollary 7. *For $0 \leq x_1, y_1, x_2, y_2, x_3, y_3 \leq m-1$ where $m \geq 3$, the 3-dimensional torus $C_m \times C_m \times C_m$ has normalized Green's function*

$$\begin{aligned}\mathcal{G}^{\times}((x_1,x_2,x_3),(y_1,y_2,y_3)) \ &= \ \frac{3}{m^2}\sum_{(j,k)\neq(0,0)} \Big[\exp\big((2\pi i j/m)(y_2-x_2)\big) \\ &\quad \exp\big((2\pi i k/m)(y_3-x_3)\big)\mathcal{G}_{(1-\cos(2\pi j/m)/2-\cos(2\pi k/m)/2)}(x_1,y_1)\Big] \\ &+ \frac{3}{m^2}\sum_{j=1}^{m}\exp\big((2\pi i k/m)(y_3-x_3)\big)\mathcal{G}_{(1-\cos(2\pi j/m))}(x_2,y_2) \\ &+ \frac{3}{m^2}\left(\frac{(m+1)(m-1)}{6m} - |y_3-x_3| + \frac{(y_3-x_3)^2}{m}\right) \\ &+ \frac{3}{m^2}\left(\frac{(m+1)(m-1)}{6m} - |y_2-x_2| + \frac{(y_2-x_2)^2}{m}\right) \\ &+ \frac{3}{m^2}\left(\frac{(m+1)(m-1)}{6m} - |y_1-x_1| + \frac{(y_1-x_1)^2}{m}\right). \end{aligned}$$

where $\mathcal{G}_\alpha$ is given in Corollary 5.

These compact formulas for Green's functions of tori offer fast alternatives to computing pseudo-inverses of their Laplacians directly. This increase in speed, essentially due to the symmetry of the torus, is reflected in $O(\log n)$ computational complexity of the Chebyshev polynomials T_n and U_n. Various algorithms for computing T_n and U_n are given in [10], and a more theoretical treatment of types of polynomials computable in $O(\log n)$ appears in [9]. The following corollary to Theorem 9 is significant because the Laplacian ($\mathcal{L}$, L, or Δ) of the torus on n vertices has rank $n-1$, and so computing its pseudo-inverse provides along the way the inverse of an $(n-1)\times(n-1)$ matrix.

Corollary 8. *Matrix pseudo-inversion of the Laplacian of the t-dimensional torus with n vertices via its Green's function is $O(t \cdot n^{2-1/t} \log n)$, provided that the matrix itself is not completely reconstructed.*

Proof. We assume the t-dimensional torus is $C_{m_1} \times \cdots \times C_{m_t}$, where $m_s \geq 3$ for $1 \leq s \leq t$ and $\prod_{s=1}^{t} m_s = n$. It suffices to compute one row of the pseudo-inverse of the normalized Laplacian in order to know the entire inverse, due to the translational symmetry of the torus; i.e., since

$$\mathbf{G}((x_1, \ldots, x_t), (y_1, \ldots, y_t)) \;=\; \mathbf{G}((0, \ldots, 0), (|y_1 - x_1|, \ldots, |y_t - x_t|)).$$

(In fact, because $|y_s - x_s|$ can be replaced by $m - |y_s - x_s| \pmod m$ without changing the value of $\mathbf{G}$, only $\prod_{s=1}^{t} \lceil m_s/2 \rceil$ of these entries must actually be computed.) Without loss of generality, $m_1 \geq \cdots \geq m_t$. Then the summation term on the right-hand side of (26) has at most $n^{1-1/t}$ summands, which can each be computed in $O(t \log n)$ time. The second and third terms can also be computed in $O(t\, n^{1-1/t} \log n)$ time, and we must compute n terms total to know all entries of the pseudo-inverse of $\mathcal{L}$. $\square$

For example, the time complexity of computing the Green's function for the 3-torus with n vertices is $O(n^{5/3} \log n)$. Such a quick pseudo-inversion formula is surprising, since matrix inversion in general has the same complexity as matrix multiplication (see [2]). Matrix multiplication is known to be $O(n^{\omega})$, where $2 \leq \omega \leq 2.376$ (see [5]). Surprisingly, for large n we can compute all of the values for the pseudo-inverse of the Laplacian for the torus faster than we can write down the whole matrix. Of course, requiring the presentation of the whole matrix rather than just the first row increases the complexity to $O(t\, n^{1-1/t} \log n + n^2)$.

6 Hitting times from Green's functions

The equivalence of Green's functions to the fundamental matrix Z of (5) under similarity transformation allows many quantities in random walks to be computed using $\mathcal{G}$. The *hitting time* $Q(x, y)$ of a simple random walk starting at vertex x with target vertex y is the expected number of steps to reach vertex y for the first time by starting at x and at each step moving to any neighbor of x with equal probability. In [3], Chung and Yao show the following relationship between $\mathcal{G}$ and Q.

Theorem 10 (Chung, Yao). *The hitting time $Q(x, y)$ satisfies*

$$Q(x, y) = \frac{vol(\Gamma)}{d_y} \mathcal{G}(y, y) - \frac{vol(\Gamma)}{\sqrt{d_x d_y}} \mathcal{G}(x, y).$$

We immediately have a computational formula for the hitting time whenever $\mathcal{G}$ is known. For instance, whenever Γ is regular with n vertices,

$$Q(x, y) = n \left(\mathcal{G}(y, y) - \mathcal{G}(x, y) \right). \tag{27}$$

Figure 1 plots hitting time in the case of $C_{49} \times C_{49}$, using Theorem 7 and (27). The domain consists of the vertices of the torus laid out in a square grid with periodic boundary, making opposite ends adjacent. The vertical axis plots the hitting time of a random walk initiated at $(0, 0)$ with target

(x, y), achieving a minimum of 0 at $(0, 0)$ and leveling off at just above 6000 steps to reach vertices farthest from the start of the walk. If (27) is computed using techniques such as Corollary 5, the hitting time expression will involve orthogonal polynomials. Aldous and Fill claim in [1] that orthogonal polynomials appear whenever the graph has sufficient symmetry, but the dependence remains largely unstudied.

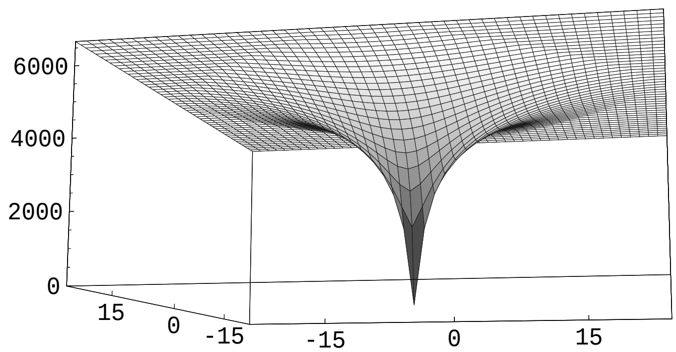

Figure 1: Graph of the hitting time of a random walk on the torus $C_{49} \times C_{49}$, laid out as a square with periodic boundary

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